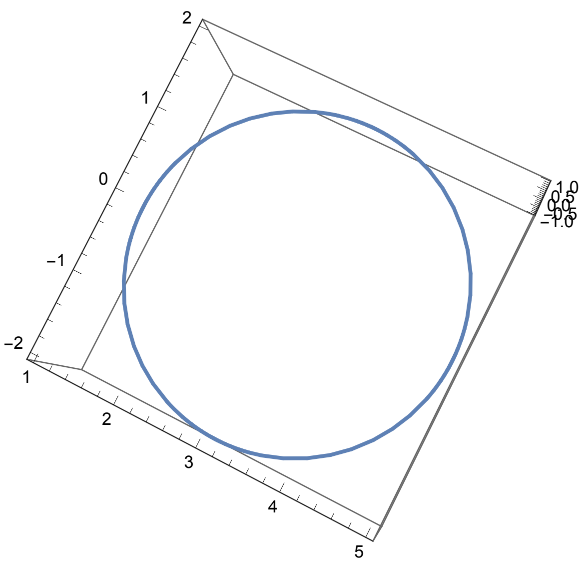
1 - 10 Parametric representations
What curves are represented by the following? Sketch them.

1. {3 + 2 Cos[t], 2 Sin[t], 0}

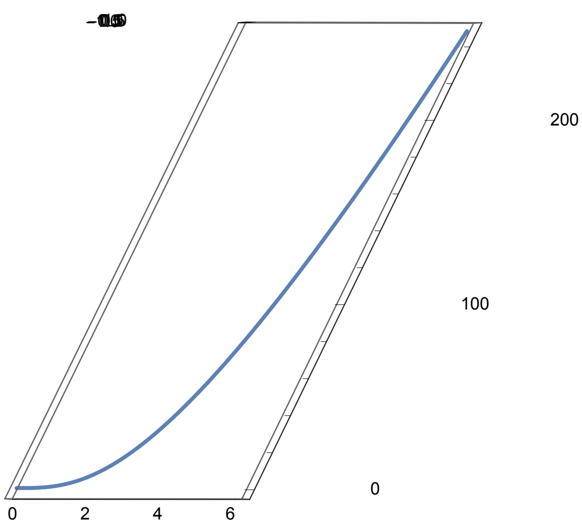
```
ParametricPlot3D[{3 + 2 Cos[t], 2 Sin[t], 0}, {t, 0, 2 π}, ImageSize → 300]
```



Above: this is a circle. Center {3, 0}, radius 2.

3. $\{0, t, t^3\}$

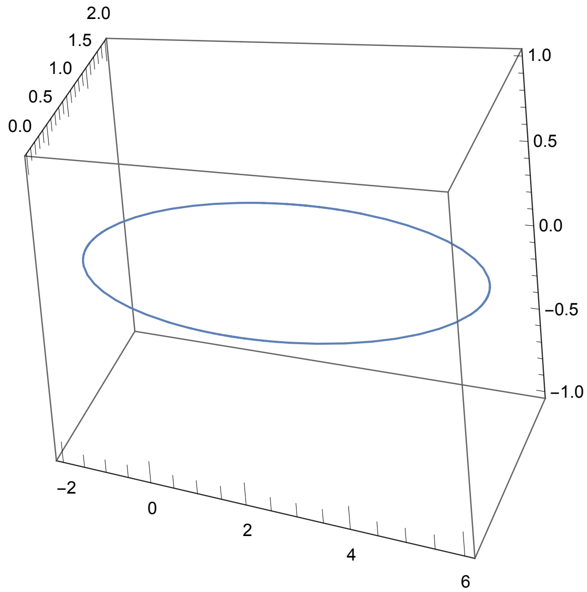
```
ParametricPlot3D[{0, t, t^3}, {t, 0, 2 π}, AspectRatio → 1, ImageSize → 300]
```



Above: this looks like half of a ‘u’ shape. The text answer calls it a cubic parabola.

5. {2 + 4 Cos[t], 1 + Sin[t], 0}

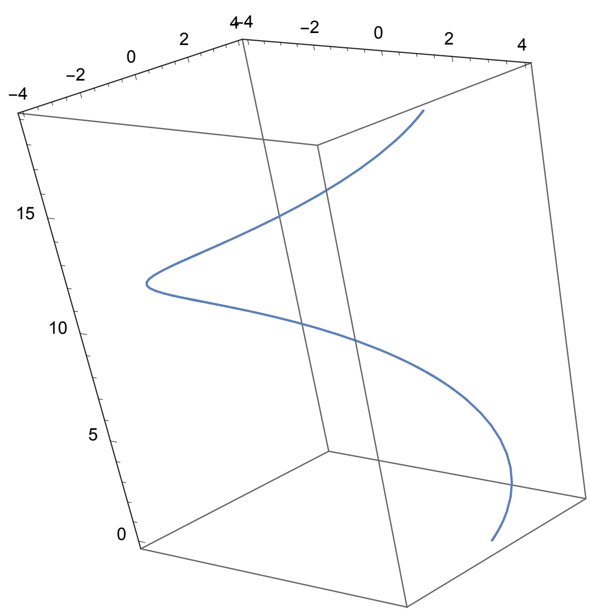
```
ParametricPlot3D[{2 + 4 Cos[t], 1 + Sin[t], 0}, {t, 0, 2 π},
 AspectRatio → 1, ImageSize → 300, PlotStyle → Thickness[0.004]]
```



Above: this looks like an ellipse.

7. {4 Cos[t], 4 Sin[t], 3 t}

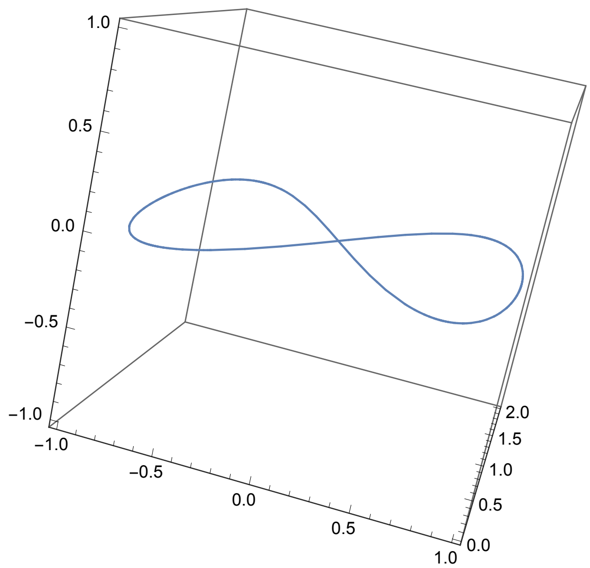
```
ParametricPlot3D[{4 Cos[t], 4 Sin[t], 3 t}, {t, 0, 2 π},
 AspectRatio → 1, ImageSize → 300, PlotStyle → Thickness[0.004]]
```



Above: this is a helix.

9. {Cos[t], Sin[2 t], 0}

```
ParametricPlot3D[{ Cos[t], 1 + Sin[2 t], 0}, {t, 0, 2 π},
 AspectRatio → 1, ImageSize → 300, PlotStyle → Thickness[0.004]]
```



Above: this is a figure-8, a “Lissajous”.

11 - 20 Find a parametric representation

11. Circle in the plane z = 1 with center {3, 2} and passing through the origin.

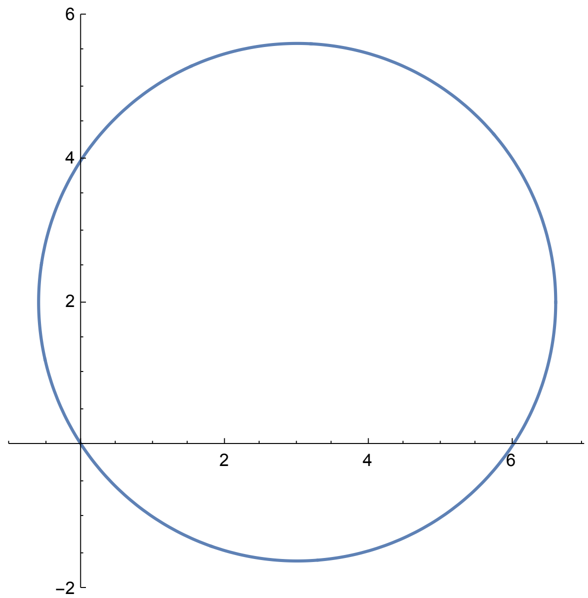
```
ClearAll["Global`*"]
```

I need the radius.

```
e1 = Norm[{3, 2}]
```

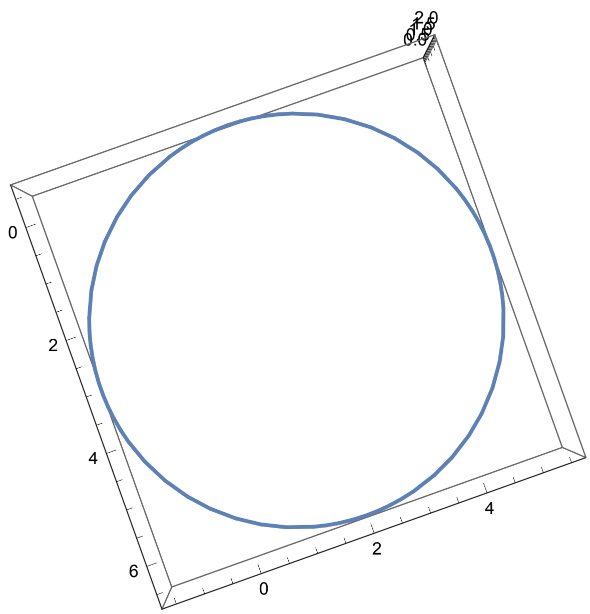
$\sqrt{13}$

```
e2 = ParametricPlot[
  {3 + √13 Cos[t], 2 + √13 Sin[t]}, {t, 0, 2 π}, ImageSize → 300]
```



The above result in 2D shows that the equation works. Only necessary to add the z-plane requirement.

```
e3 = ParametricPlot3D[
  {3 + √13 Cos[t], 2 + √13 Sin[t], 1}, {t, 0, 2 π}, ImageSize → 300]
```



With the pseudo-parallax effect, it is hard to tell whether the origin is part of the circle.

```
e4 = Solve[3 + √13 Cos[t] == 0 && 2 + √13 Sin[t] == 0]
```

$\{\{t \to \text{ConditionalExpression}[-\pi + \text{ArcTan}[\frac{2}{3}] + 2\pi C[1], C[1] \in \text{Integers}]\}\}$

```
e5 = N[π + ArcTan[2/3]]
```

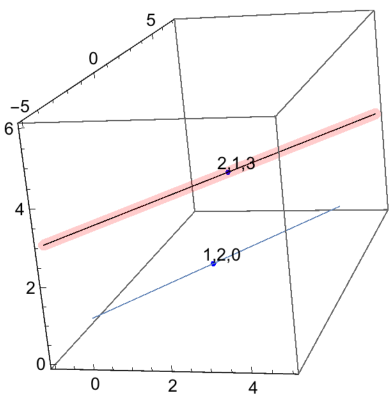
3.7296

Since this result points to a number in the defining interval of the function, I take it to show that (0,0,1) is in the circle.

13. Straight line through {2, 1, 3} in the direction of **i** + 2**j**.

```
ClearAll["Global`*"]
```

```
Show[ParametricPlot3D[{u + 2, 2 u + 1, 3}, {u, -3, 3},
  PlotStyle → {Red, Thickness[0.03], Opacity[.2]}, ImageSize → 200],
 ParametricPlot3D[{t + 2, 2 t + 1, 3}, {t, -3, 3},
  PlotStyle → {Black, Thickness[0.003]}], ParametricPlot3D[
  {t + 1, 2 t + 2, 0}, {t, -3, 3}, PlotStyle → {Teal, Thickness[0.003]}],
 ListPointPlot3D[{{2, 1, 3}, {1, 2, 0}}, PlotStyle → Blue], Graphics3D[
  {Text["2,1,3", {2.2, 1.2, 3.2}], Text["1,2,0", {1.2, 2.2, .2}]}]]
```



The red line goes through the specified point and has the same direction as [1,2,0]. The text answer line (black) runs inside the red line.

15. Straight line y = 4x -1, z = 5x.

```
ClearAll["Global`*"]
```

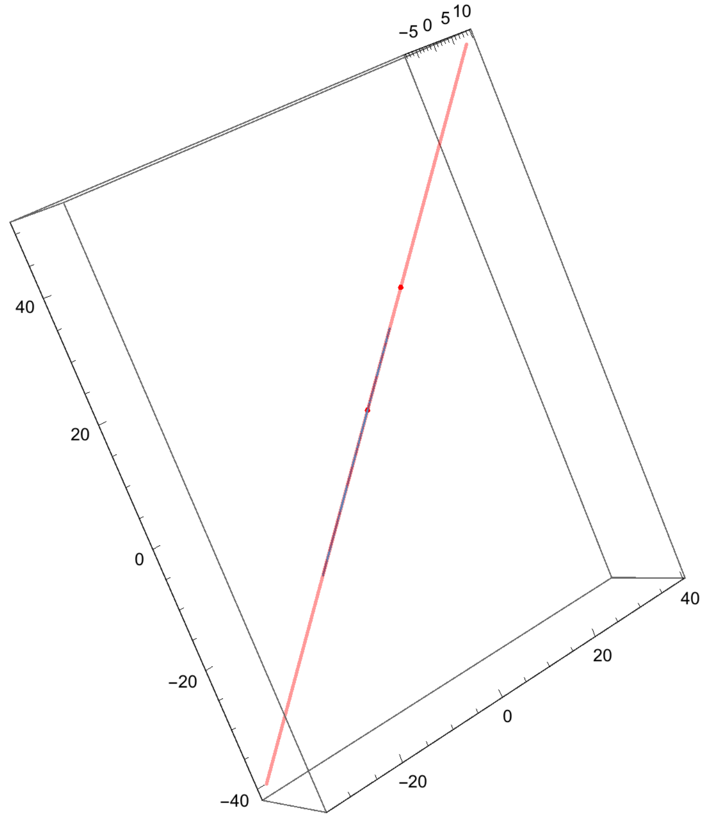
```
Solve[y == 4 x - 1 && z == 5 x && x == 1]
```

{{x → 1, y → 3, z → 5}}

```
Solve[y == 4 x - 1 && z == 5 x && x == 4]
```

{{x → 4, y → 15, z → 20}}

```
e1 = Show[ParametricPlot3D[{3 u + 1, 12 u + 3, 15 u + 5}, {u, -3, 3},
   PlotStyle → {Red, Thickness[0.005], Opacity[.4]}], ParametricPlot3D[
   {t, 4 t - 1, 5 t}, {t, -3, 3}, PlotStyle → Thickness[0.003]],
  ListPointPlot3D[{{1, 3, 5}, {4, 15, 20}}, PlotStyle → Red]]
```



Above: the line shown meets the requirements. The text answer line is shown within.

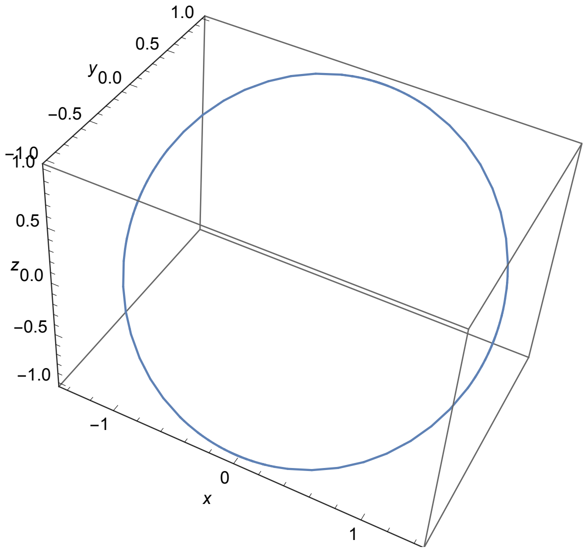
17. Circle $\frac{1}{2}x^2 + y^2 = 1$, z = y.

```
ClearAll["Global`*"]
```

This didn't look like a circle when I first did the problem. This problem is treated in the s.m., so I take that general direction. It looks like an ellipse, and the form of the equation can be changed.

e1 = $\frac{x^2}{\left(\sqrt{2}\right)^2} + y^2 = 1$. From the general form, it can be seen that it is an ellipse with semi-major axis of $\sqrt{2}$. Putting that into parametric form would be $\left(\sqrt{2}\cos u + a\right) + (\sin u + b)$, where the a and b are center locations. Here both are zero.

```
ParametricPlot3D[{√2 Cos[u], Sin[u], Sin[u]}, {u, 0, 2 Pi},
 AxesLabel → {x, y, z}, PlotStyle → Thickness[0.004], ImageSize → 300]
```



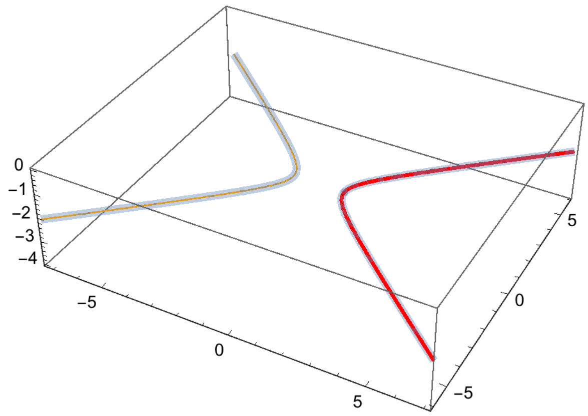
The z part of the equation just mirrors y, it’s not necessary to ponder what effect it might have. But after it is plotted, it can be seen to be a true circle, due to that z component, which represents its tilt away from the xy-plane.

19. Hyperbola $4x^2$ - $3y^2$, z = -2

```
ClearAll["Global`*"]
```

Hyperbola. I looked this one up before. The parametric version is $\frac{a}{\cos t}$, $b \tan t$. In this case $a = 1$, $b = -\frac{\sqrt{3}}{2}$.

```
Show[ParametricPlot3D[{1/Cos[u], -(√3/2) Tan[u], -2},
  {u, 0, 2 π}, Exclusions → {Cos[u] == 0}, ImageSize → 300,
  PlotStyle → {Thickness[0.015], Opacity[.4]}], ParametricPlot3D[
  {{Cosh[t], (√3/2) Sinh[t], -2}, {-Cosh[t], (√3/2) Sinh[t], -2}},
  {t, -2 π, 2 π}, ImageSize → 300, PlotStyle → {Red, Thickness[0.003]}]]
```



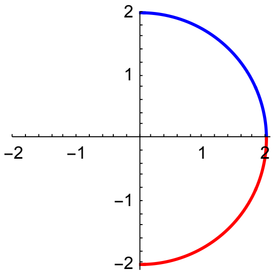
Three things going on here. The first is my own plot of the hyperbola, skinny black. The second and third are the fatter versions of the hyperbola, but only the red one is contained in the text answer. It seems deficient to me because it is necessary to show two functions in order to get both sides of the hyperbola.

21. Orientation. Explain why setting t = -t* reverses the orientation of {a Cos[t], a Sin[t], 0}.

This follows example 1 on p. 382. The answer to the reversed orientation is simply the way the functions work. Plotting a small segment of the example function,

```
p1 = ParametricPlot[{2 Cos[t], 2 Sin[t]}, {t, 0, π/2}, ImageSize → 140,
   AspectRatio → 1, PlotStyle → Blue, PlotRange → {{-2, 2.1}, {-2.1, 2}}];
p2 = ParametricPlot[{2 Cos[-t], 2 Sin[-t]}, {t, 0, π/2},
   ImageSize → 140, AspectRatio → 1, PlotStyle → Red];
```

```
Show[{p1, p2}]
```



Obviously, trig functions are sensitive to signs.

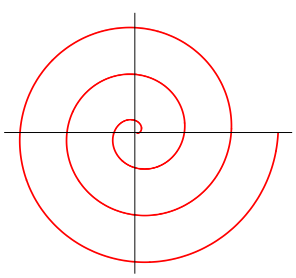
23. CAS project. Famous curves in polar form. Use your CAS to graph the following curves given in polar form $\rho = \rho(\theta)$, $\rho^2 = x^2 + y^2$, $\text{Tan}[\theta] = \frac{y}{x}$, and investigate their form depending on parameters a and b.

$\rho = a\,\theta$	Spiral of Archimedes
$\rho = a\,e^{b\,\theta}$	Logarithmic spiral
$\rho = \frac{2\,a\,\text{Sin}[\theta]^2}{\text{Cos}[\theta]}$	Cissoid of Diocles
$\rho = \frac{a}{\text{Cos}[\theta]} + b$	Conchoid of Nicomedes
$\rho = \frac{a}{\theta}$	Hyperbolic spiral
$\rho = \frac{3\,a\,\text{Sin}[2\,\theta]}{\text{Cos}[\theta]^3+\text{Sin}[\theta]^3}$	Folium of Descartes
$\rho = 2\,a\,\frac{\text{Sin}[3\,\theta]}{\text{Sin}[2\,\theta]}$	Maclaurin ' s trisectrix
$\rho = 2\,a\,\text{Cos}[\theta] + b$	Pascal ' s snail

(a) Spiral of Archimedes

Plot found on *MathWorld*.

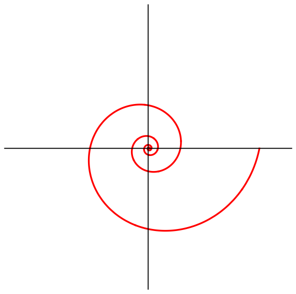
```
ParametricPlot[t {Cos[t], Sin[t]}, {t, 0, 6 Pi}, Ticks → None,
 PlotStyle → {Red, Thickness[0.006]}, ImageSize → 150]
```



(b) Logarithmic spiral.

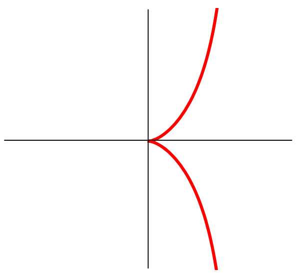
Plot found on MathWorld.

```
PolarPlot[ⅇ^(.2 t), {t, 0, 8 π}, PlotStyle → {Red, Thickness[0.006]},
 Ticks → None, PlotRange → All, ImageSize → 150]
```



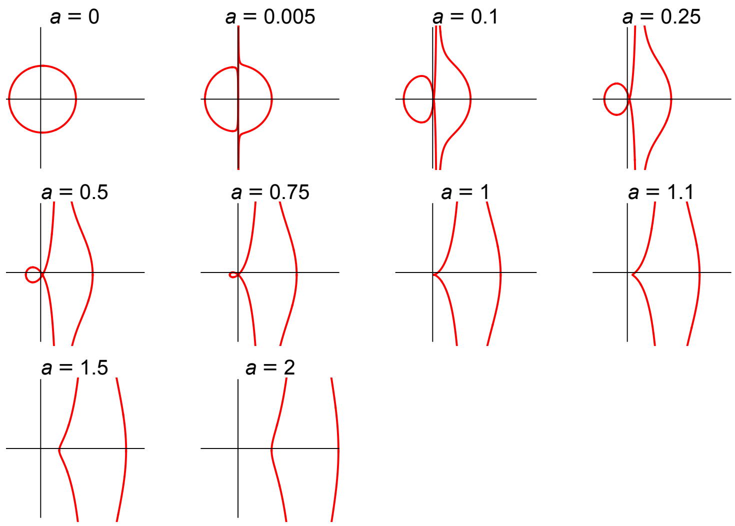
(c) Cissoid of Diocles
Plot found on MathWorld.

```
ParametricPlot[{2 Sin[t]^2, (2 Sin[t]^3)/Cos[t]}, {t, -1.5, 1.5},
 PlotRange → {-3, 3}, Ticks → None, PlotStyle → Red, ImageSize → 150]
```



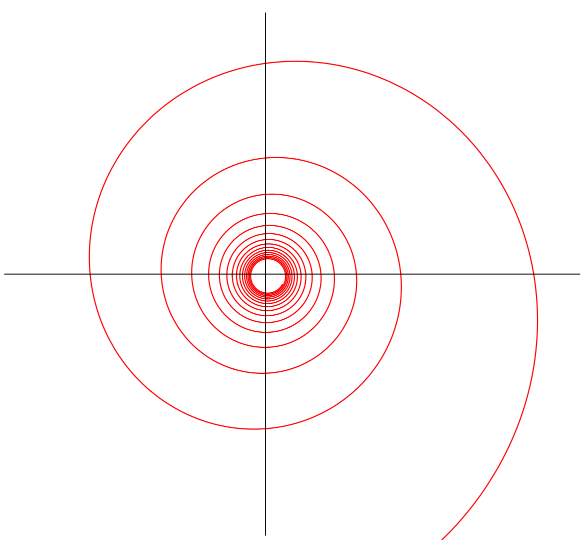
(d) Conchoid of Nicomedes
Plot found on MathWorld.

```
GraphicsGrid[Partition[With[{eps = 10^-5}, Function[a, Show[
      PolarPlot[1 + a Sec[θ], Evaluate[{θ, #[[1]] + eps, #[[2]] - eps}],
         PlotStyle → {Red, Thickness[0.014]}] & /@ Partition[
        Range[0, 2, 1 / 2] π, 2, 1], PlotRange → {{-1, 3}, {-2, 2}},
      Ticks → None, PlotLabel → ToExpression["a"] ==
        (a /. HoldPattern[Rational[x__]] :→ InlineFraction[x])]]] /@
    {0, .005, .1, .25, .5, .75, 1, 1.1, 1.5, 2}
  ], 4, 4, {1, 1}, {}], ImageSize → 400, AspectRatio → 0.6]
```



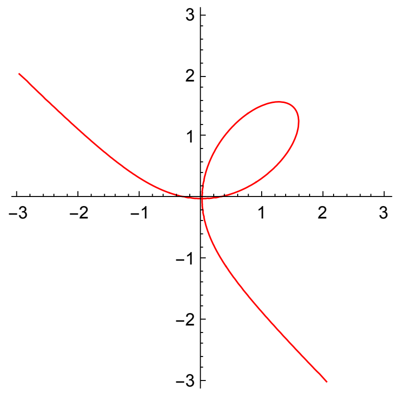
(e) Hyperbolic spiral
Plot found on MathWorld.

```
PolarPlot[1 / θ, {θ, 0.001, 100}, PlotStyle → {Red, Thickness[0.002]},
 Ticks → None, ImageSize → 300, PlotRange → {{-.15, 0.18}, {-0.15, 0.15}}]
```



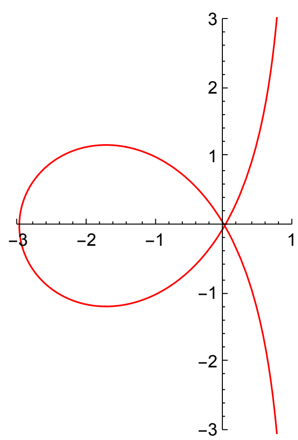
(f) Folium of Descartes
Plot found on MathWorld.

```
ContourPlot[x^3 + y^3 == 3 x y, {x, -3, 3}, {y, -3, 3},
 ContourStyle → {Red, Thickness[0.004]}, AspectRatio → Automatic,
 Frame → False, Axes → True, ImageSize → 200]
```



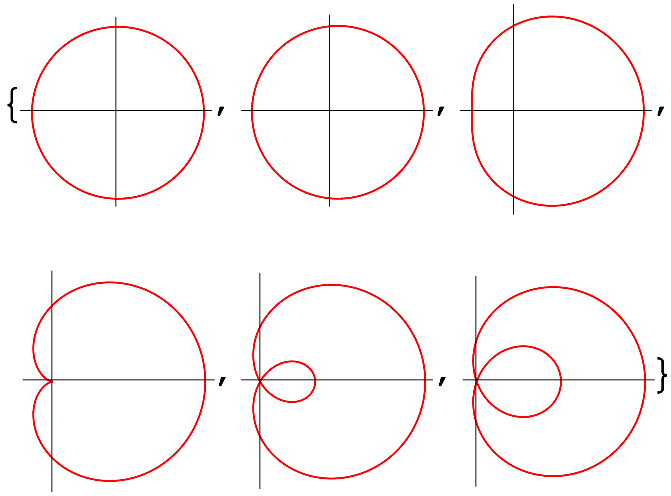
(g) Maclaurin' s trisectrix
Plot found on MathWorld.

```
ParametricPlot[{1, t} (t^2 - 3) / (t^2 + 1), {t, -4, 4},
 AspectRatio → Automatic, PlotRange → {{-3, 1}, {-3, 3}},
 PlotStyle → {Red, Thickness[0.006]}, ImageSize → 150]
```



(h) Pascal' s snail
Plot found on MathWorld.

```
ParametricPlot[(1 + # Cos[t]) {Cos[t], Sin[t]}, {t, 0, 2 π},
   ImageSize → 100, PlotStyle → {Red, Thickness[0.01]},
   Ticks -> None] & /@ {0, .1, .5, 1, 2, 3}
```



24 - 28 Tangent

Given a curve C: **r**[t], find a tangent vector ***r***'[*t*], a unit tangent vector ***u***'[*t*], and the tangent of *C* at *P*. Sketch curve and tangent.

25. r[t] = {10 Cos[t], 1, 10 Sin[t]}, P : {6, 1, 8}

```
ClearAll["Global`*"]
```

```
e1 = Solve[10 Cos[t] == 6 && 10 Sin[t] == 8]
```

$\{\{t \to \text{ConditionalExpression}[\text{ArcTan}[\frac{4}{3}] + 2\pi C[1], C[1] \in \text{Integers}]\}\}$

```
e2 = e1[[1, 1, 2, 1]]
```

$\text{ArcTan}[\frac{4}{3}] + 2\pi C[1]$

```
e3 = e2 /. C[1] → 0
```

$\text{ArcTan}[\frac{4}{3}]$

Above: this is the value that satisfies the problem vector function for the given point.

```
e4 = r[t_] = {10 Cos[t], 1, 10 Sin[t]}
```

```
{10 Cos[t], 1, 10 Sin[t]}
```

$\text{e5} = \text{r}[\text{ArcTan}[\frac{4}{3}]]$

```
{6, 1, 8}
```

The above shows that P corresponds to t = ArcTan$\left[\frac{4}{3}\right]$ with regard to the given function **r**.

```
e6 = r'[ArcTan[4/3]]
```

```
{-8, 0, 6}
```

```
e7 = e5 + w e6
```

```
{6 - 8 w, 1, 8 + 6 w}
```

Above: this is the tangent of C:r[t] at P, called Q(w).

Below: using the (8) on p. 384 of the text, I find the unit tangent at P,

```
e8 = r'[t]
{-10 Sin[t], 0, 10 Cos[t]}
```

```
e9 = Norm[e8]
```

$\sqrt{100\,\text{Abs}[\text{Cos}[t]]^2 + 100\,\text{Abs}[\text{Sin}[t]]^2}$

```
e10 = FullSimplify[e9]
```

$10\sqrt{\text{Abs}[\text{Cos}[t]]^2 + \text{Abs}[\text{Sin}[t]]^2}$

$\text{e11} = 10\ /.\ \sqrt{\text{Abs}[\text{Cos}[t]]^2 + \text{Abs}[\text{Sin}[t]]^2} \to 1$

```
10
```

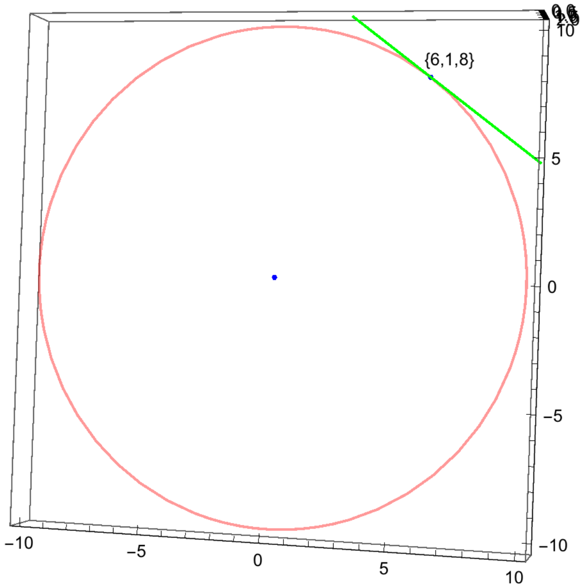
$\text{e12} = \text{e8}\,\frac{1}{\text{e11}}$

```
{-Sin[t], 0, Cos[t]}
```

Above: this is the unit tangent. The above answers agree with the text.

```
e14 = e12[{6, 1, 8}]
{-Sin[t], 0, Cos[t]}[{6, 1, 8}]
```

```
e13 = Show[ParametricPlot3D[{10 Cos[t], 1, 10 Sin[t]}, {t, -3.275, 3},
    PlotStyle → {Red, Thickness[0.005], Opacity[.4]}, ImageSize → 300],
  ListPointPlot3D[{{6, 1, 8}, {0, 1, 0}}, PlotStyle → Blue],
  ParametricPlot3D[{6 - 8 w, 1, 8 + 6 w}, {w, -1, 1},
    PlotStyle → {Green, Thickness[0.005]}],
  Graphics3D[{Text["{6,1,8}", {6.7, 1, 8.7}]}]]
```



27. $r[t] = \left\{t, \frac{1}{t}, 0\right\}$, P : $\left\{2, \frac{1}{2}, 0\right\}$

```
ClearAll["Global`*"]
```

By inspection it can be seen that P represents t=2, as demonstrated in e2.

e1 = r[t_] = $\left\{t, \frac{1}{t}, 0\right\}$

$\left\{t, \frac{1}{t}, 0\right\}$

e2 = r[2]

$\left\{2, \frac{1}{2}, 0\right\}$

e3 = r'[t]

$\left\{1, -\frac{1}{t^2}, 0\right\}$

e4 = r'[2]

$\left\{1, -\frac{1}{4}, 0\right\}$

```
e5 = e2 + w e4
```

$$\left\{2 + w, \frac{1}{2} - \frac{w}{4}, 0\right\}$$

Above: the tangent of C:r[t], matching the answer in the text.

```
e9 = Norm[e3]
```

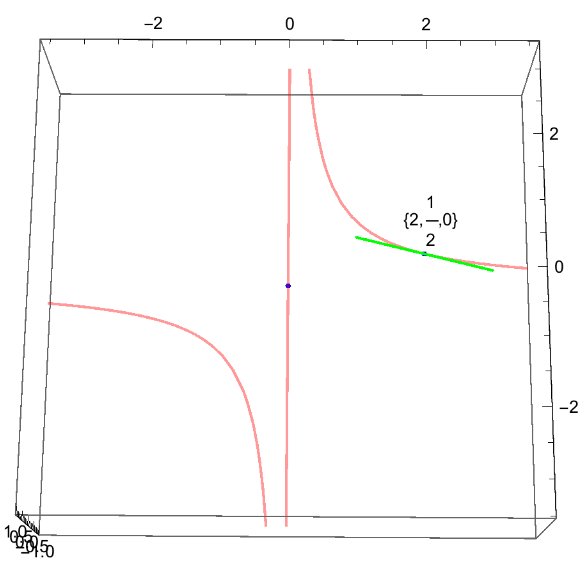
$$\sqrt{1 + \frac{1}{\text{Abs}[t]^4}}$$

$$e10 = \frac{e3}{e9}$$

$$\left\{\frac{1}{\sqrt{1 + \frac{1}{\text{Abs}[t]^4}}}, -\frac{1}{t^2 \sqrt{1 + \frac{1}{\text{Abs}[t]^4}}}, 0\right\}$$

Above: this is the unit tangent.

```
e13 = Show[ParametricPlot3D[{t, 1/t, 0}, {t, -3.5, 3.5},
    PlotStyle → {Red, Thickness[0.005], Opacity[.4]}, ImageSize → 300],
  ListPointPlot3D[{{2, 1/2, 0}, {0, 0, 0}}, PlotStyle → Blue],
  ParametricPlot3D[{2 + w, 1/2 - w/4, 0}, {w, -1, 1},
    PlotStyle → {Green, Thickness[0.005]}],
  Graphics3D[{Text["{2, 1/2, 0}", {2.1, 1, 0}]}]]
```



29 - 32 Length

Find the length and sketch the curve.

29. Catenary **r**[t] = {t, Cosh[t]} from t = 0 to t = 1.

```
ClearAll["Global`*"]
```

```
e1 = r[t_] = {t, Cosh[t]}
```

```
{t, Cosh[t]}
```

```
e2 = r'[t]
```

```
{1, Sinh[t]}
```

```
e3 = len = Integrate[√(r'[t].r'[t]), {t, 0, 1}]
```

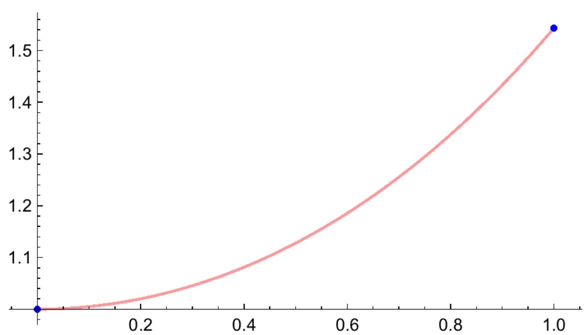
```
Sinh[1]
```

```
e4 = N[Sinh[1]]
```

```
1.1752
```

Above in green: two values which agree with the answer in the text.

```
Show[ParametricPlot[{t, Cosh[t]}, {t, 0, 1},
  PlotStyle → {Red, Thickness[0.005], Opacity[.4]}, ImageSize → 300],
 ListPlot[{{0, Cosh[0]}, {1, Cosh[1]}}, PlotStyle → Blue],
 AxesOrigin → Automatic]
```



31. Circle r[t] = {a Cos[t], a Sin[t]} from {a, 0} to {0, a}.

```
ClearAll["Global`*"]
```

```
e1 = r[t_] = {a Cos[t], a Sin[t]}
```

```
{a Cos[t], a Sin[t]}
```

```
e2 = r'[t]
```

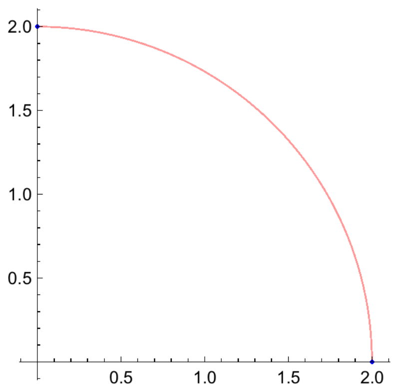
```
{-a Sin[t], a Cos[t]}
```

```
e3 = len = Integrate[√(r'[t].r'[t]), {t, 0, π/2}]
```

$$\frac{\sqrt{a^2}\,\pi}{2}$$

Above: The answer shown agrees with the text. Integration limits based on initial values.

```
Show[ParametricPlot[{2 Cos[t], 2 Sin[t]}, {t, 0, π/2},
  PlotStyle → {Red, Thickness[0.005], Opacity[.4]}, ImageSize → 200],
 ListPlot[{{2 Cos[0], 2 Sin[0]}, {2 Cos[π/2], 2 Sin[π/2]}}, PlotStyle → Blue],
 AxesOrigin → Automatic]
```



33. Plane curve. Show that numbered line (10) on p. 385 implies $\ell = \int_a^b 1 + (y')^2 \, dx$ for the length of a plane curve C: y = f[x], z = 0, and a = x = b.

I should show the numbered line (10) referred to in this problem

$$\ell = \int_a^b \sqrt{r'.r'}\, dt$$

The symbol ℓ refers to the length of a line.

35 - 46 Curves in mechanics

Forces acting on moving objects (cars, airplanes, ships, etc.) require the engineer to know corresponding ***tangential*** and ***normal accelerations***. In problems 35 - 38 find them, along with the ***velocity*** and ***speed***. Sketch the path.

35. Parabola **r**[t] = {t, t^2, 0}. Find **v** and **a**.

```
ClearAll["Global`*"]
```

```
e1 = rr[t_] = {t, t², 0}
```

$\{t, t^2, 0\}$

```
e2 = rr'[t]
```

```
{1, 2 t, 0}
```

```
e3 = rr''[t]
```

```
{0, 2, 0}
```

Above: general acceleration.

```
e4 = v = Norm[rr'[t]]
```

$$\sqrt{1 + 4\,\text{Abs}[t]^2}$$

Above: magnitude of the velocity.

$$\text{e5} = \text{aT} = \frac{\text{e2.e3}}{\text{e4}}$$

$$\frac{4\,t}{\sqrt{1 + 4\,\text{Abs}[t]^2}}$$

Above: the tangential acceleration.

$$\text{e6} = \text{aN} = \frac{\text{Norm}[\text{Cross}[\text{e2}, \text{e3}]]}{\text{Norm}[\text{e2}]}$$

$$\frac{2}{\sqrt{1 + 4\,\text{Abs}[t]^2}}$$

Above: the normal acceleration.

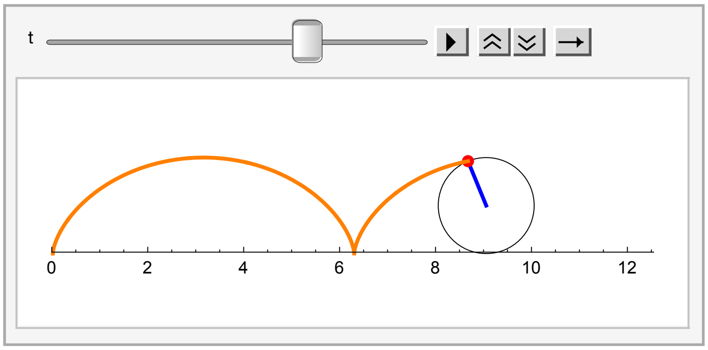
Green cells above match the answer in the text. The formulas used here were used in my workthru of Ed9, and originally came from Larson p. 816.

37. Cycloid **r**[t] = (R Sin[ω t]+R t)**i** +(R Cos[ω t] + R)**j**. This is the path of a point on the rim of a wheel of radius R that rolls without slipping along the x-axis. Find **v** and **a** at the maximum y-values of the curve.

```
ClearAll["Global`*"]
```

First I'll throw in an animation I found at *https://mathematica.stackexchange.com/questions/8832/-making-mathematical-animations-with-mathematica*, which is surprisingly compact.

```
Animate[Show[Graphics[
   Translate[Rotate[{Circle[], Thick, Blue, Line[{{0, 0}, {0, -1}}],
      Red, PointSize[.02], Point[{0, -1}]}, -t], {t, 0}],
   PlotRange → {{0, 4 Pi}, {-2, 2}}, ImageSize → {Large, Tiny},
   Axes → {True, False}, AxesOrigin → {0, -1}],
  ParametricPlot[{(a - Sin[a]), (-Cos[a])}, {a, 0, t},
   PlotStyle → Directive[Thick, Orange]]],
 {t, 0.001, 4 Pi}, AnimationRunning → False]
```



The site *https://math.mit.edu/~mckernan/Teaching/12-13/Autumn/18.02/l_6.pdf* give the formulas for velocity, speed, and acceleration along the cycloid path. The video at *https://www.youtube.-com/watch?v=xwkz1-8nxDc* shows the scaled velocity vector in an animation. The velocity:

```
vel[t_] = {1 - Cos[t], Sin[t]}
{1 - Cos[t], Sin[t]}
```

reaches a max at t=π and t=3π, which is situated at the max value of y, the problem criterion.

```
FindMaximum[Norm[vel[t]], {t, 0}]
{2., {t → 9.42478}}
```

The speed there is the same as the norm of the velocity vector.

```
cycspeed[t_] = √2 (1 - Cos[t])^(1/2)
√2 √(1 - Cos[t])
```

```
FindMaximum[cycspeed[t], {t, 0}]
{2., {t → 9.42478}}
```

```
FindMaximum[x Cos[x], {x, 2}]
{0.561096, {x → 0.860334}}
```

The acceleration

```
cycaccel[t_] = {Sin[t], Cos[t]}
{Sin[t], Cos[t]}
```

seems to be constant. (Only shows here at the path beginning because that was the guess, I think.)

```
FindMaximum[Norm[cycaccel[t]], {t, 0}]
```

FindMaximum::fmgz : Encountered a gradient that is effectively zero. The result returned may not be a maximum; it may be a minimum or a saddle point. ≫

```
{1., {t → 0.}}
```

Since the problem asked for the acceleration at the top of the path,

```
cycaccel[9.42477796554220 1`]
```

$\{-4.77282\times 10^{-9}, -1.\}$

where the magnitude is

```
Norm[%]
1.
```

Looking now at the text answer, I see it does not seem to answer the questions asked in the problem description. However, I can interpret it to mean that not t = 0 is intended, but t=max-y-value, which is t = odd*n*π. For my part, I have to make adjustments for the ω, and the R, which I ignored before.

```
velta[t_] = {R (1 - Cos[ω t]), R (Sin[ω t])}
{R (1 - Cos[t ω]), R Sin[t ω]}
```

For some reason Mathematica will not simplify the result of the following, which would otherwise match the text answer.

```
velta[π]
{R (1 - Cos[π ω]), R Sin[π ω]}
```

```
velta'[t]
{R ω Sin[t ω], R ω Cos[t ω]}
```

```
velta'[π]
{R ω Sin[π ω], R ω Cos[π ω]}
```

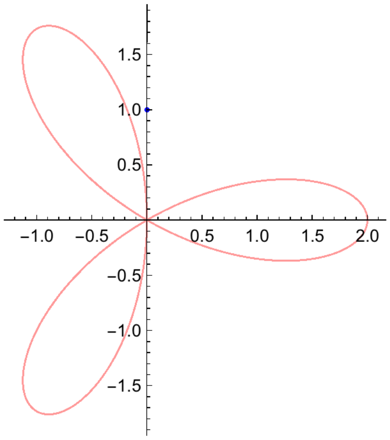
The Mathematica answers match the text answers, although *Mathematica* will not simplify them. For info, text shows $\mathbf{v}(0)=(\omega+1)R\,\mathbf{i}$ and $\mathbf{a}(0) = -\omega^2 R\,\mathbf{j}$ as answers.

39 - 42 The use of a CAS may greatly facilitate the investigation of more complicated paths, as they occur in gear transmissions and other constructions. To grasp the idea, using a CAS, graph the path and find velocity, speed, and tangential and normal acceleration.

39. **r**[t] = {Cos[t] + Cos[2t], Sin[t] - Sin[2t]}

```
ClearAll["Global`*"]
```

```
e13 =
 Show[ParametricPlot[{Cos[t] + Cos[2 t], Sin[t] - Sin[2 t]}, {t, -2 π, 2 π},
   PlotStyle → {Red, Thickness[0.005], Opacity[.4]}, ImageSize → 200],
  ListPlot[{{6, 1}, {0, 1}}, PlotStyle → Blue] (*,ParametricPlot[
   {6-8 w,1,8+6 w},{w,-1,1},PlotStyle→{Green,Thickness[0.005]}],
  Graphics3D[{Text["{6,1,8}",{6.7,1,8.7}]}]*)]
```




```
e1 = positionfunction = r[t_] = {Cos[t] + Cos[2 t], Sin[t] - Sin[2 t]}
```

{Cos[t] + Cos[2 t], Sin[t] - Sin[2 t]}

```
e2 = velocityfunction = r'[t]
```

{-Sin[t] - 2 Sin[2 t], Cos[t] - 2 Cos[2 t]}

```
e3 = accelerationfunction = r''[t]
```

{-Cos[t] - 4 Cos[2 t], -Sin[t] + 4 Sin[2 t]}

Above: general acceleration.

```
e4 = magnitudeofvelocity = Norm[r'[t]]
```

$\sqrt{\text{Abs}[\text{Cos}[t] - 2\,\text{Cos}[2\,t]]^2 + \text{Abs}[-\text{Sin}[t] - 2\,\text{Sin}[2\,t]]^2}$

Above: magnitude of the velocity.

```
e41 = velocitymagnitudesquared = e4^2
```

$\text{Abs}[\text{Cos}[t] - 2\,\text{Cos}[2\,t]]^2 + \text{Abs}[-\text{Sin}[t] - 2\,\text{Sin}[2\,t]]^2$

```
e42 =
 FullSimplify[(Cos[t] - 2 Cos[2 t])^2 + (-Sin[t] - 2 Sin[2 t])^2 == 5 - 4 Cos[3 t]]
```

True

```
e43 = 5 - 4 Cos[3 t]
```

```
5 - 4 Cos[3 t]
```

Above: Mathematica verifies that the text answer for $|v|^2$ agrees with the tangled forest above. (I did remove the absolute value restrictions for the test, since they did not show up in the text answer.)

```
e5 = aT = e2.e3/e43
```

$$\frac{1}{5 - 4 \text{Cos}[3 t]}((-\text{Cos}[t] - 4 \text{Cos}[2 t]) (-\text{Sin}[t] - 2 \text{Sin}[2 t]) + (\text{Cos}[t] - 2 \text{Cos}[2 t]) (-\text{Sin}[t] + 4 \text{Sin}[2 t]))$$

```
e6 = FullSimplify[e5]
```

$$\frac{6 \text{Sin}[3 t]}{5 - 4 \text{Cos}[3 t]}$$

Above: this would be the tangential acceleration, except it needs a **v**.

prime dot doubleprime divided by magnitude of velocity.

```
e8 = tangentialacceleration = e6 e2
```

$$\left\{\frac{6 (-\text{Sin}[t] - 2 \text{Sin}[2 t]) \text{Sin}[3 t]}{5 - 4 \text{Cos}[3 t]}, \frac{6 (\text{Cos}[t] - 2 \text{Cos}[2 t]) \text{Sin}[3 t]}{5 - 4 \text{Cos}[3 t]}\right\}$$

Above: the actual tangential acceleration, but more complicated than the text answer.

```
e9 = aN = Norm[Cross[e2, e3]]/Norm[e2]
```

Cross::nonn1 : The arguments are expected to be vectors of equal length, and the number of arguments is expected to be 1 less than their length ≫

$$\text{Norm}[\{-\text{Sin}[t] - 2 \text{Sin}[2 t], \text{Cos}[t] - 2 \text{Cos}[2 t]\} \times \{-\text{Cos}[t] - 4 \text{Cos}[2 t], -\text{Sin}[t] + 4 \text{Sin}[2 t]\}] \Big/ \left(\sqrt{\text{Abs}[\text{Cos}[t] - 2 \text{Cos}[2 t]]^2 + \text{Abs}[-\text{Sin}[t] - 2 \text{Sin}[2 t]]^2}\right)$$

Above, oops, this is only 2 dimensional. How to get normal acceleration? Looks like I need $\frac{du}{ds}\left(\frac{ds}{dt}\right)^2$ where u is the unit tangent vector and s is the speed, or norm of velocity. And $u = \frac{r'[t]}{|r'[t]|}$.

```
e10 = u = e2/e4
```

$$\left\{\frac{-\text{Sin}[t]-2\,\text{Sin}[2\,t]}{\sqrt{\text{Abs}[\text{Cos}[t]-2\,\text{Cos}[2\,t]]^2+\text{Abs}[-\text{Sin}[t]-2\,\text{Sin}[2\,t]]^2}},\ \frac{\text{Cos}[t]-2\,\text{Cos}[2\,t]}{\sqrt{\text{Abs}[\text{Cos}[t]-2\,\text{Cos}[2\,t]]^2+\text{Abs}[-\text{Sin}[t]-2\,\text{Sin}[2\,t]]^2}}\right\}$$

Above: hold on, looks like it's getting complicated. The text mentions an easier way: normal acceleration is general acceleration minus tangential acceleration.

```
e11 = e3 - e6
```

$$\left\{-\text{Cos}[t]-4\,\text{Cos}[2\,t]-\frac{6\,\text{Sin}[3\,t]}{\sqrt{\text{Abs}[\text{Cos}[t]-2\,\text{Cos}[2\,t]]^2+\text{Abs}[\text{Sin}[t]+4\,\text{Cos}[t]\,\text{Sin}[t]]^2}},\ -\text{Sin}[t]+4\,\text{Sin}[2\,t]-\frac{6\,\text{Sin}[3\,t]}{\sqrt{\text{Abs}[\text{Cos}[t]-2\,\text{Cos}[2\,t]]^2+\text{Abs}[\text{Sin}[t]+4\,\text{Cos}[t]\,\text{Sin}[t]]^2}}\right\}$$

Above: this would be the normal acceleration. How to simplify? I notice that the text does not bother to report the normal acceleration, so I could just skip it.

```
e12 = FullSimplify[(Cos[t] - 2 Cos[2 t])^2 + (Sin[t] + 4 Cos[t] Sin[t])^2]
```

$$5-4\,\text{Cos}[3\,t]$$

```
e13 = normalacceleration =
  e11 /. (Abs[Cos[t] - 2 Cos[2 t]]^2 + Abs[Sin[t] + 4 Cos[t] Sin[t]]^2) ->
    5 - 4 Cos[3 t]
```

$$\left\{-\text{Cos}[t]-4\,\text{Cos}[2\,t]-\frac{6\,\text{Sin}[3\,t]}{\sqrt{5-4\,\text{Cos}[3\,t]}},\ -\text{Sin}[t]+4\,\text{Sin}[2\,t]-\frac{6\,\text{Sin}[3\,t]}{\sqrt{5-4\,\text{Cos}[3\,t]}}\right\}$$

Above: that looks quite a bit better. However, I don't see a path to anything simpler.

41. **r**[t] = {Cos[t], Sin[2t], Cos[2t]}

```
ClearAll["Global`*"]
```

```
e1 = positionfunction = r[t_] = {Cos[t], Sin[2 t], Cos[2 t]}
```

```
{Cos[t], Sin[2 t], Cos[2 t]}
```

```
e2 = velocityfunction = r'[t]
```

```
{-Sin[t], 2 Cos[2 t], -2 Sin[2 t]}
```

```
e3 = accelerationfunction = r''[t]
```

```
{-Cos[t], -4 Sin[2 t], -4 Cos[2 t]}
```

Above: general acceleration.

```
e4 = magnitudeofvelocity = Norm[r'[t]]
```

$$\sqrt{4\,\text{Abs}[\text{Cos}[2\,t]]^2 + \text{Abs}[\text{Sin}[t]]^2 + 4\,\text{Abs}[\text{Sin}[2\,t]]^2}$$

Above: magnitude of the velocity.

$$\text{e5} = \text{squareofvelocity} = \text{e4}^2$$

$$4\,\text{Abs}[\text{Cos}[2\,t]]^2 + \text{Abs}[\text{Sin}[t]]^2 + 4\,\text{Abs}[\text{Sin}[2\,t]]^2$$

$$\text{e6} = \text{e5} \;/.\; \left(4\,\text{Abs}[\text{Cos}[2\,t]]^2 + \text{Abs}[\text{Sin}[t]]^2 + 4\,\text{Abs}[\text{Sin}[2\,t]]^2\right) \to 4 + \text{Sin}[t]^2$$

$$4 + \text{Sin}[t]^2$$

Above: hand simplification.

$$\text{e6} = \frac{\text{e2.e3}}{\text{e6}}$$

$$\frac{\text{Cos}[t]\,\text{Sin}[t]}{4 + \text{Sin}[t]^2}$$

Above: tangential acceleration, but it is incomplete, because it needs to be multiplied by **v**.

```
e7 = e6 e2
```

$$\left\{-\frac{\text{Cos}[t]\,\text{Sin}[t]^2}{4 + \text{Sin}[t]^2},\ \frac{2\,\text{Cos}[t]\,\text{Cos}[2\,t]\,\text{Sin}[t]}{4 + \text{Sin}[t]^2},\ -\frac{2\,\text{Cos}[t]\,\text{Sin}[t]\,\text{Sin}[2\,t]}{4 + \text{Sin}[t]^2}\right\}$$

$$\text{e7} = \text{normalacceleration} = \frac{\text{Norm}[\text{Cross}[\text{e2}, \text{e3}]]}{\text{Norm}[\text{e2}]}$$

$$\left(\sqrt{\left(\text{Abs}[-4\,\text{Cos}[2\,t]\,\text{Sin}[t] + 2\,\text{Cos}[t]\,\text{Sin}[2\,t]]^2 + \text{Abs}[2\,\text{Cos}[t]\,\text{Cos}[2\,t] + 4\,\text{Sin}[t]\,\text{Sin}[2\,t]]^2 + \text{Abs}\left[-8\,\text{Cos}[2\,t]^2 - 8\,\text{Sin}[2\,t]^2\right]^2\right)}\right) \Big/ \left(\sqrt{4\,\text{Abs}[\text{Cos}[2\,t]]^2 + \text{Abs}[\text{Sin}[t]]^2 + 4\,\text{Abs}[\text{Sin}[2\,t]]^2}\right)$$

Again, normal acceleration is a mess. Try it the other way.

```
e8 = normalacceleration = e3 - e6
```

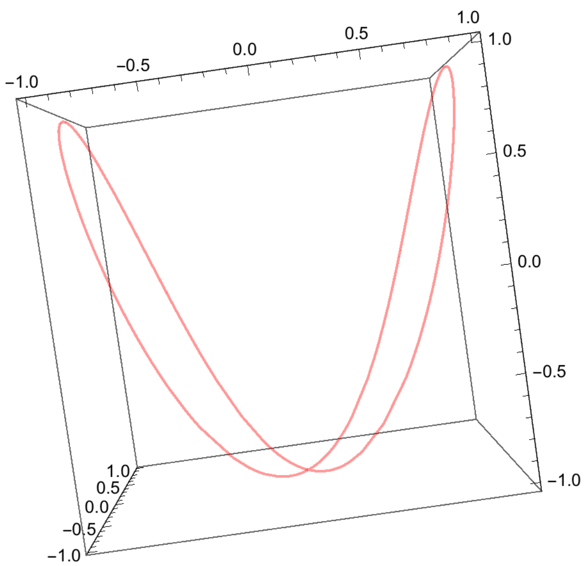
$$\left\{-\text{Cos}[t] - \frac{\text{Cos}[t]\,\text{Sin}[t]}{4+\text{Sin}[t]^2},\ -\frac{\text{Cos}[t]\,\text{Sin}[t]}{4+\text{Sin}[t]^2} - 4\,\text{Sin}[2\,t],\ -4\,\text{Cos}[2\,t] - \frac{\text{Cos}[t]\,\text{Sin}[t]}{4+\text{Sin}[t]^2}\right\}$$

```
e9 = FullSimplify[e8]
```

$$\left\{\text{Cos}[t]\left(-1+\frac{2\,\text{Sin}[t]}{-9+\text{Cos}[2\,t]}\right),\ \left(-4+\frac{1}{-9+\text{Cos}[2\,t]}\right)\text{Sin}[2\,t],\ -4\,\text{Cos}[2\,t]+\frac{\text{Sin}[2\,t]}{-9+\text{Cos}[2\,t]}\right\}$$

Above: I don't see much to choose between e8 and e9.

```
e10 = ParametricPlot3D[{Cos[t], Sin[2 t], Cos[2 t]}, {t, 0, 2 π},
  PlotStyle → {Red, Thickness[0.005], Opacity[.4]}, ImageSize → 300]
```



43. Sun and Earth. Find the acceleration of the Earth toward the sun from numbered line (19) on p. 387 and the fact that Earth revolves about the sun in a nearly circular orbit with an almost constant speed of 30 km/s.

```
ClearAll["Global`*"]
```

Centripetal acceration is $\frac{v^2}{r}$, so the first thing to try is $\frac{900}{\text{distance–to–sun}}$

```
Table[N[900/(n * 10^8)], {n, {1.46, 1.52}}]
```

$\{6.16438\times10^{-6},\ 5.92105\times10^{-6}\}$

The text answer says $5.98*10^{-6}$ which is in the above range, based on max to min distance, found on *https://www.windows2universe.org/?page=/earth/statistics.html*, which is why I award a green here. Anyway, the acceleration works out to be in km/sec^2.

45. Satellite. Find the speed of an artificial Earth satellite traveling at an altitude of 80 miles above Earth's surface, where g = 31 ft/sec^2. (The radius of the Earth is 3960 miles.)

```
ClearAll["Global`*"]
```

Using the same formula as above, $\frac{v^2}{r} = a_{\text{centripetal}}$

```
Solve[v^2/(3960 * 5280) == 31, v]
```

$\{\{v \to -2640\sqrt{93}\}, \{v \to 2640\sqrt{93}\}\}$

```
N[2640 Sqrt[93]]
```

25 459.2

The above yellow does not match exactly with the text answer, which prompts the test below.

```
N[Solve[(25 700)^2/(3960 * 5280) == a, a]]
```

{{a → 31.5891}}

I think the above shows that the given values in the problem were not observed by the text answer.

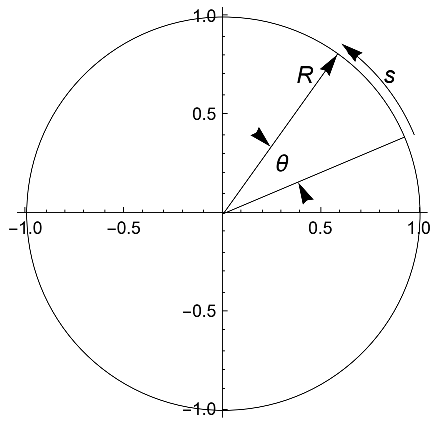
47 - 55 Curvature and torsion

47. Circle. Show that a circle of radius a has a curvature $1/a$.

```
ClearAll["Global`*"]
```

The concept of curvature revolves around the idea of how much a curve "bends" for each unit of advance, s. The definition in the text is on p. 389. The idea of a center of curvature is not necessary for the text, only the idea of a unit tangent, and the rate of change of the tangent is judged for the degree of curvature. I'm going with a different idea, as described on *https://www.solitaryroad.com/c361.html* and at other places, and shown in the figure below. A unit circle. A full trip around is worth 2πR in terms of s, and the 2π can be fractionated, so that for any advance, Δs = R$\Delta\theta$. This simple model boils down to

$$\frac{\text{accumulated angle}}{\text{accumulated advance}} = \frac{\Delta\theta}{\Delta s} = \frac{\Delta\theta}{R\Delta\theta} = \frac{1}{R}$$



See problem 49 for more on curvature.

49. Plane curve. Using numbered line (22*) on p. 389, show that for a curve y = f[x]

$\kappa[x] = \frac{|y''|}{\left(1+(y')^2\right)^{3/2}}$ where $\left(y' = \frac{dy}{dx},\ \text{etc.}\right)$

(Note: Problem 49 calls for reference to numbered line (22*), but no such numbered line exists in the present section. Numbered line (22) looks like it deals with related matter and it reads: $\kappa(s) = |u'(s)| = |r''(s)|$)

```
ClearAll["Global`*"]
```

The equation in the problem is the one shown as the formula for curvature on many sites, for example *https://www.intmath.com/applications-differentiation/8-radius-curvature.php.* Trying to keep it as simple as possible, the demonstration begins with the statement that the curvature is equal to

$$K = \text{Limit}\left[\frac{\Delta\theta}{\Delta s},\ \theta \to 0\right] = \frac{d\theta}{ds}$$

the chain rule enters here, as

$$\frac{d\theta}{ds} = \frac{d\theta}{dx}\frac{dx}{ds}$$

and so the two derivatives on the right have to be accounted for. Noting that

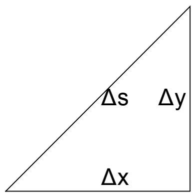
$$\text{Tan}[\theta] = \frac{dy}{dx} \Rightarrow \theta = \text{ArcTan}\left[\frac{dy}{dx}\right] \Rightarrow \frac{d\theta}{dx} = \frac{d}{dx}\text{ArcTan}\left[\frac{dy}{dx}\right]$$

The last term can be handled by Mathematica without sweat,

```
D[ArcTan[y'[x]], x]
```

$$\frac{y''[x]}{1 + y'[x]^2}$$

It will still be necessary to find $\frac{dx}{ds}$. Scrounging around on the level of tiny limits makes two things possible. First, Δs can be treated as a straight line. Second, derivatives can be treated like fractions. So in considering the following diagram, where s is a portion of the curve I am investigating, I will have



$$\Delta s^2 = \Delta x^2 + \Delta y^2 \;\Rightarrow\; \frac{\Delta s^2}{\Delta x^2} = \frac{\Delta x^2}{\Delta x^2} + \frac{\Delta y^2}{\Delta x^2} \;\Rightarrow\; \frac{\Delta s^2}{\Delta x^2} = 1 + \frac{\Delta y^2}{\Delta x^2} \;\Rightarrow\; \frac{\Delta s}{\Delta x} = \sqrt{1 + \frac{\Delta y^2}{\Delta x^2}} \;\Rightarrow\; \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

and finally

$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + (y'[x])^2}}$$

So that,

$$K = \frac{y''[x]}{1 + y'[x]^2} * \frac{1}{\sqrt{1 + (y'[x])^2}}$$

$$\frac{y''[x]}{\left(1 + y'[x]^2\right)^{3/2}}$$

Demonstrating what was intended. However, the numerator is not protected against negative values, as the problem description suggests. I did not see this refinement on any of the sites I browsed. Reading over this site: *https://math.stackexchange.com/questions/2118029/what-is-the-meaning-of-second-derivative/2118081*, I did not get the idea that the formula couldn't work with a negative second derivative of y, but I may have missed something. Problem 49 did say to use numbered line (22*), and if it meant instead to use (22), that would constrain the result to the absolute value of the second derivative.